2. THE IMPOSSIBLE

§2.1. Nothing is Impossible!?

It's impossible! It can never be done! Dangerous words! How often has the short-sightedness of man placed limits on what can be achieved?

Man will never fly in a heavier-than-air machine and certainly will never stand on the moon. Total 'impossibilities' yet we've seen them come about. Computers will never be able to play a game of chess to grand-master standard. Yet it has happened.

But of course some things are eternally impossible. As children we grappled with the idea of the impossible.

"Bet there's nothing God can't do."

"Bet there is."

"What, then? Bet you three marbles you can't think of something God can't do."

"He can't make 2 plus 2 make 5."

"Yeah, but that's not possible. I mean God can do anything that's *logically* possible."

"So he could lift up the world?"

"Sure, he could even lift up the sun with his little finger!"

"Well then, is he able to find something so big he can't lift it?"

"But that's impossible."

"No it's not. I can easily find something so big that I can't lift it, so why can't God?"

Now of course there's nothing impossible about something being logically impossible. We can all make up problems that have no solutions. And if a problem is impossible it's important to know that, otherwise we can waste a lot of time.

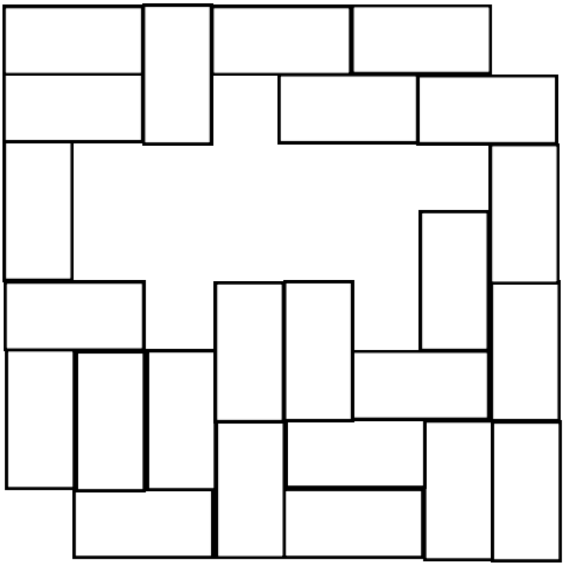
This book is about a lot of impossible things. The more important of them have helped to delineate the boundaries of rational thought. Because they involve things close to the limits of human reasoning we may from time to time look over the fence into philosophy, but our feet will stay firmly on the side of logical thought as we dabble in the mathematics at the edge of the rational universe.

§2.2. The Domino Puzzle

How are we able to prove that a problem has no solution? The most obvious way is to check through, and eliminate, every possibility. But how can we prove that something is impossible if there are an enormous number of possibilities? Get a computer to do the checking? But what if there are *infinitely* many possibilities?

The following puzzle involves a huge, but finite, number of possibilities, though it could easily be adapted to one with an infinite number.

Take 31 dominos and place them on a chessboard (each one covering two adjacent squares) so that the two squares that remain uncovered are at diagonally opposite corners. At first sight the following appears to be well on the way to a solution but closer inspection reveals that it can’t be completed.



That doesn't prove it *is* impossible of course. It may simply mean we started wrongly. But what an enormous number of possible ways of starting we'd need to check!

We might try unsuccessfully for quite some time and declare in disgust that "it's impossible" but the nagging thought would remain, "maybe just one more try will do it".

Yet it is indeed impossible. You can take that as a challenge if you wish, but you're really wasting your time. It's known to be impossible, not because many have tried and failed and not because a computer has worked through every conceivable possibility. It has been *proved* to be impossible. And the proof involves a clever but exceedingly simple idea.

Use a chessboard pattern of black and white squares. Each domino must, of course, cover one black square and one white one. The 31 dominos therefore cover the same number of black and white squares and so the two remaining squares must be of *opposite* colours. But diagonally opposite corners of a chessboard, as every chess player knows, have the *same* colour. A contradiction is reached if a solution were to exist. So of course no solution could possibly exist.

§2.3. Proof By Contradiction

Not every proof of impossibility is as short and transparent as this one. But they all rely on the simple idea that any assumption that leads to a contradiction must be false.

We begin by assuming that whatever we're trying to prove impossible is in fact *possible*. We then attempt to use logic to reach a contradiction, that is, something which is both true and false. If we succeed in producing this nonsense we know that our assumption of possibility must be false and we'll have proven impossibility.

Some people get worried about the validity of this type of reasoning. "You can't make assumptions in proofs." It's true that if you're allowed to assume that what you're trying to prove true *is* true, then naturally you'll succeed all the time, no matter what you're trying to prove.

Assume that the moon is made of green cheese. Therefore if you land on the moon and dig up a sample, it will be green in colour, and will have a strong cheesy flavour. Therefore the sample will be green cheese. And if you repeat this experiment at numerous other locations on the moon you must get the same result. Hence the moon *is* made of green cheese! But of course that proves nothing. We might get away with such a fallacy if our chain of arguments is so long that our listener forgets what we'd assumed in the first place. But fallacious reasoning it is, nevertheless.

So of course it *is* fallacious to assume what you're trying to prove. But that's not what we're doing in a proof by contradiction. In such a proof we're assuming that what we're trying to prove is *false*, or that the so-called impossible is in fact possible. And that's a totally different thing.

Proof by contradiction is not some esoteric rule thought up by logicians or mathematicians. It's just ordinary common sense that we use all the time. "You couldn't have put the milk away because it's still on the bench." Analysing the logic behind this assertion we find that it's a proof by contradiction.

Theorem: You didn't put the milk away.
Proof: Suppose that you *did* put the milk away.
Then the milk is in the refrigerator.
[Here there's the unspoken assumption that no-one else has been around to take it out again.]
But the milk is still on the bench and so is not in the fridge.
[That we are talking about the same bottle is another unspoken assumption.]

Contradiction! Therefore you did not put the milk away.

You, the accused, might still dispute this argument. Nothing in life is quite as clear-cut as in mathematics. But the only way you could validly attack it would be to draw out and dispute one or other of these unspoken assumptions. The underlying logic of the argument itself is perfectly sound.

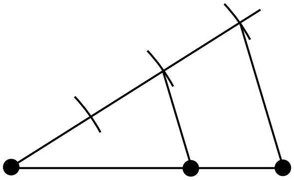
Impossibilities are everywhere, not just at the edges of rational thought. Before we journey out to the uttermost parts of the rational universe we'll look at a number of other perfectly ordinary impossibilities. Some are quite famous in the history of mathematics. Others are mere curiosities. Our purpose in examining them is to help us feel quite at home with proof by contradiction because that's the tool we'll need on our journey.

§2.4. The Square Root of 2 is Irrational

An irrational number isn't one which is crazy. It simply means one which cannot be expressed exactly as a *ratio* of two whole numbers, like 2/3 or 22/7. Ratios have a geometric significance. The Greeks were able to divide any given line in any ratio of two whole numbers.

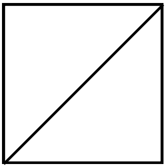
For example to find the point which is two thirds of the way along a given line segment, you construct a second line from one end of the first and mark off three equal lengths (with a compass, of course — using rulers to measure was considered unacceptable).

The third point is joined to the other end of the original line segment and other lines are drawn parallel to it as in the following diagram. If you remember from school how to draw parallel lines using ruler and compass, well and good. If you've forgotten, it doesn't matter. There *is* a way.



This can be easily adapted to construct a line segment that is any rational multiple of the one given.

But the Greeks soon learnt of a theorem that's associated with the name Pythagoras. The square on the hypotenuse is equal to the sum of the squares on the other two sides. Construct a square (which the Greeks could do easily, using their rulers and compasses) and by this famous theorem the length of the diagonal of the square must be $\sqrt{2}$ times the length of the side. (Diagonal squared $= 1^2 + 1^2 = 2$.)



Since the only numbers that existed at that time, or rather the only numbers that had been invented, were rational numbers, it was obvious that √2 had to be the ratio of two integers. It simply remained to find the two integers.

The rational number 10/7, when squared, gives 2.04..., which is close to, but not exactly equal to, 2. A better approximation is given by 1393/985. Its square is 1.9999989... very much closer. Try 8119/5741.

Then came the embarrassing truth. They discovered an argument that demonstrated the impossibility of finding such integers. No rational number gives exactly 2 when squared. So here was a line which had no length! That can't be! To get around this difficulty, new numbers had to be invented.

Theorem: √2 is irrational
Proof: Suppose that, on the contrary, it is rational. (Here's a classic Proof By Contradiction.)
Let *m* and *n* be the two whole numbers whose ratio, when squared, gives exactly 2.
Then $\left(\frac{m}{n}\right)^2 = 2$.
But this means that $\frac{m^2}{n^2} = 2$ and so $m^2 = 2n^2$, that is, m^2 must be exactly twice as big as n^2.

Now consider the number of factors of 2 which divide these numbers. However many factors of 2 there are that divide n, clearly exactly double that number divide n^2. In fact the number of factors of 2 in any perfect square, n^2 or m^2, must be even. But that means an odd number of factors of 2 divides $2n^2$ and an even number dividing m^2. This can’t happen if they’re equal!

This contradiction rests firmly on a single assumption – that $\sqrt{2}$ is rational. This assumption cannot stand. The square root of 2 must be irrational.

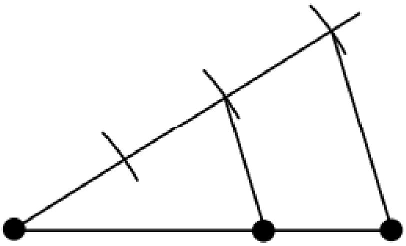
§2.5. It is Impossible to Trisect a Given Angle by Ruler and Compass

One of the famous classical impossibilities concerns ruler and compass constructions. This type of geometric construction was a highly developed art form in the time of the ancient Greeks because for them, arithmetic was built on the foundation of geometry. Ruler and compass construction was as important a tool then as the calculator is today.

The ruler wasn’t used to measure lengths. In fact any straight edge would do. What the Greeks had against measurement was that it wasn’t exact. No matter how fine the divisions, a length may fall between two of them and the human eye is called upon to estimate.

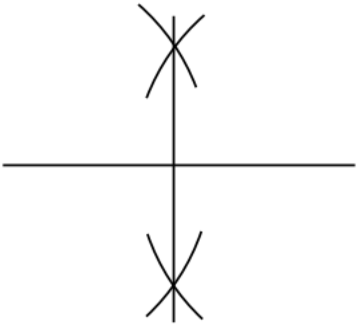
The Greeks were intoxicated by perfection. Any method had to be theoretically exact. And this they could achieve with straight-edge and compass – at least for many problems.

They could, for example, bisect any given angle. Most school pupils learn how to do this. With the compass point on the vertex of the angle,draw an arc cutting the two arms of the arc at points A, B. Now with any convenient radius (but the same for each) draw intersecting arcs, one with A as centre and one with centre at B. Joining the intersection of these arcs to the vertex of the angle exactly bisects the angle.



The method is mathematically exact. Using theorems of congruent triangles one can prove that the two angles created at the vertex are equal, each exactly half the original. Of course to do it in practice, no matter how carefully you carry out the construction, all sorts of little errors creep in. But the method is *mathematically* exact.

Lines of any given length can be bisected by a similar construction. Here the centres of the arcs are the endpoints of the line.



What was really tantalising was that although lines can be trisected (3 equal pieces) there appeared to be no method for trisecting *angles*. This really disturbed them because it was obvious to them that it could be done. Why should there be any difference? Aren’t lengths and angles just different geometric manifestations of the same numbers?

Much effort went into looking for such a construction without success. All was wasted effort. Many, many centuries passed before a proof that such a construction is impossible was discovered. It’s too technical to present here, but it’s worth pointing out that, like many proofs of impossibility, the breakthrough came by cleverly converting the problem to one involving whole numbers.

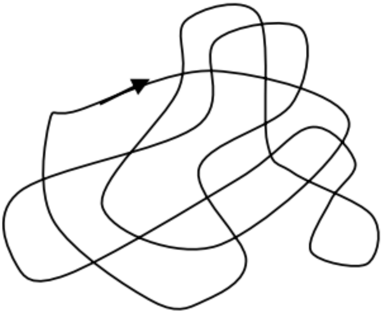
In the trisection case, there’s a number which can be associated with any ruler and compass construction called the “degree of the corresponding field extension”.

Never mind what that means. Suffice to point out that it starts at 1 and with each stage in a ruler and compass construction it either remains the same or it doubles. So only exact powers of 2 are possible: 1, 2, 4, 8, 16, ...

But it can be shown that a method which trisects a 60 degree angle, must be capable of producing a field extension whose degree is exactly 3. Clearly 3 is not a power of 2 and so we get a contradiction if we assume that angle trisection is always possible.

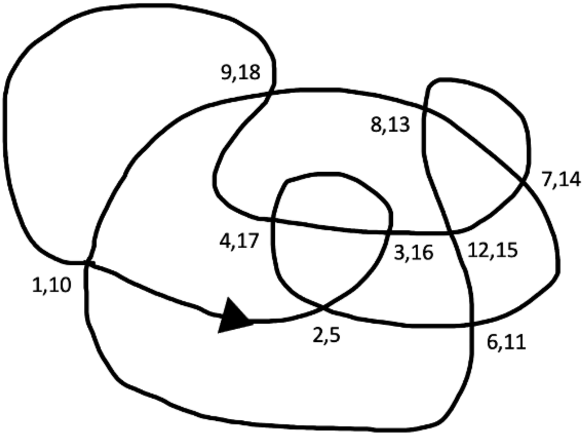
§2.6. Scribbles

Draw a **scribble**. By this I mean a continuous line which crosses itself many times and ends up where it starts. Oh, and you are not allowed to pass through a previous crossing.



You’ll have a number of crossings where two parts of the scribble cross over. Start at any crossing you like and number them in order: 1,2,3, ... When you revisit a crossing you must give it a second number. Continue until all crossings have been given two numbers.

Now a **double crossing** is one where one of its two numbers is double the other and a **triple crossing** is one where one of its numbers is three times the other,. There's no difficulty in producing double crossings. This scribble has two of them: 7, 14 and 9, 18. But there are no triple crossings.



Can we create a scribble which includes at least one triple crossing? It might have to be an exceedingly complicated scribble with millions of crossings, one of which might be a crossing labelled as (123123, 369369).

The problem can’t be solved. There is no solution. You might like to try to find one just to get the ‘feel’ of it, but don't try too hard because the puzzle is really quite impossible. But how can we be sure of this? After all there’s no limit to the complexity of the scribble so it’s just not possible to check all cases.

§2.7. Why No Triple Crossings?

Suppose it *can* be done. (Notice that we’ve started in the usual way for a Proof by Contradiction.) Then we’d have a (k, $3k$) crossing somewhere. We visit when the count is k and revisit when the count has reached $3k$.

Question: How many times will we pass a crossing between these two visits?

Answer: An odd number of times.

This might seem wrong because the difference between k and $3k$ is $2k$ which is even. But think again. How many numbers are there between 4 and 12? Do you think there are 8, because 12 – 4 = 8? No, there are only 7 numbers between 4 and 12. They are 5, 6, 7, 8, 9, 10 and 11. When we subtract 4 from 12 and get 8 we’re counting the one-unit sections between 4 and 8. But the number of dividing points is never equal to the number of sections. It’s always one more if we’re counting both ends and or one less if we’re not (as in this case).

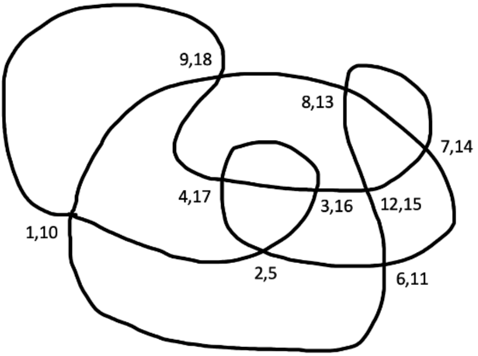
	4		5		6		7		8		9		10		11		12
sections		1		2		3		4		5		6		7		8	
intervening nos.			1		2		3		4		5		6		7		

Always, between the two visits to a (*k*, 3*k*) triple crossing, there are 2*k* sections and so 2*k* –1 visits to other crossings. So the answer to the question is “an odd number of times”.

Second Answer: An even number of times.

Before you start pointing out that this contradicts what we concluded earlier, consider the supporting argument.

If you start at any crossing in a scribble and move around till you revisit that crossing you’ll have traced out a smaller scribble. Those parts you haven’t traced will also be a smaller scribble. What you’ll have done is to decompose the original scribble into two simpler ones, linked at the crossing you started with.



You can think of one of these smaller scribbles as being the boundary of a region and the other scribble as being a closed path (‘closed’ here just means that it ends where it starts) which cuts across the first scribble in a number of places. Now because of the principle that “what goes in must come out” (this must hold because the scribbles don’t have any free ends), the two smaller scribbles must cut each other in an *even* number of places.

So when you go from a (k, $3k$) triple crossing at visit k, until you revisit it at visit $3k$, you’ll have passed through an *even* number of crossings.

But wait a minute, we've overlooked places where the scribble that’s traced out on this journey may cross itself. As well as the even number of places where the two scribbles cross we must add the places where the first scribble crosses itself. And couldn’t that be an odd number?

Yes, of course. In the above example the solid line and the dotted line are the two smaller scribbles that together make up the whole scribble. Now you can see that the solid line cuts itself just once. So doesn’t that destroy the evenness of the number of intervening crossings?

Not at all. Remember, we’re not counting crossings but visits. So, when the scribble we’re following crosses itself, that counts as *two* visits. Including these selfcrossings merely adds an even number to an already even number.

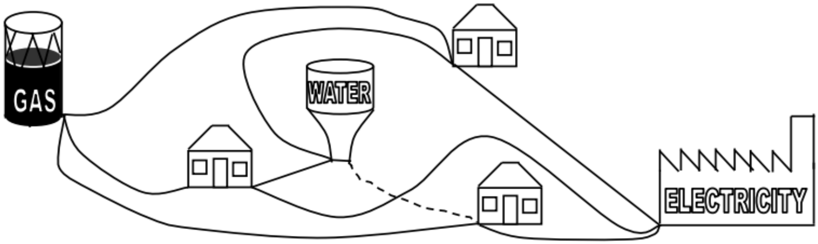
So for these reasons the number of visits to crossings between the first and second visits to our mythical triple crossing is even.

The fact that we previously convinced ourselves that this number is odd, and the fact that no number can be simultaneously odd and even, completes the argument. If a triple crossing were to exist then we'd have a contradiction, and once even a single contradiction is allowed to creep in, others follow: odd = even , true = false, black = white and the whole edifice of knowledge crumbles to dust.

You may be getting the impression that the difference between odd and even is at the heart of every proof of impossibility. This is certainly true in many cases. However we'll now see a few examples where the impossibility uses other methods.

§2.8. The Utilities Puzzle

Imagine that you have three houses, each of which has to be connected to the three utilities of gas, water and electricity. Now the catch is this. Pipes and wires are not allowed to cross over one another. Why this should be is never properly explained in the puzzle. Perhaps the world is a sort of two-dimensional 'Flatland'.



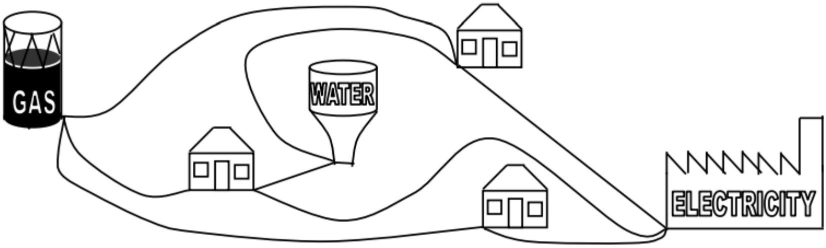
It's very easy to get a solution that almost works, where only one pipe or wire remains to be installed. But no amount of ingenuity can come up with one that works completely. Try as you might, no matter how ingenious and how contorted you make the routes, nothing seems to work.

Now you might be a pragmatist and conclude, after a bit of fruitless experimentation, that it's impossible. Or you might accept the authority of the experts and say that if nobody has managed to find a solution after all these years then it must indeed be impossible.

But if you're happy to leave it there then you don't possess mathematical curiosity. "Perhaps one day, a solution might be found." Unless the impossibility has been ruled out by a water-tight logical argument the problem would continue to tantalise mathematicians. But just such an argument *has* been found and a proof of impossibility can produce as much excitement in a mathematician as a solution would have.

The key to proving the impossibility of solving the Utilities Puzzle lies in counting. We suppose that a solution exists and count the number of points (well that's easy – there are 6 points, 3 houses plus 3 utilities), the number of connecting lines (that's easy too – there's a pipe or wire from each of the 3 houses to each of the 3 utilities, that's 9 altogether) and the number of regions enclosed by the lines.

For example if we take the above attempt at a solution and remove the incomplete pipe from the bottom house to the waterworks the number of regions, including the outside, is four.



How many regions will there be in a solution to the puzzle? We might say that there would be 5, one more than in the above because a ninth line would split one region into two. But remember that there's no way of successfully putting in a ninth line to the above picture. If there *is* a solution we'd have to start from scratch.

So how on earth can we count the number of regions until we've drawn the picture? And if the problem is impossible we can never draw the picture. Ah, but there’s another, sneakier, way to do this.

You see, there’s a connection between these three numbers which holds for any map on a plane surface. It is called Euler's Formula:

$$\mathbf{V + F - E = 2}$$

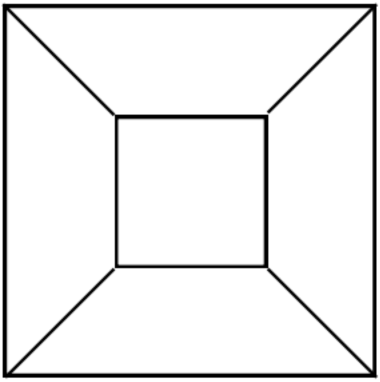
Here V is the number of ‘vertices’ (that just means points), F is the number of ‘faces’ (that just means regions) and E is the number of ‘edges’ (or connecting lines).

Usually this formula is quoted for solid figures, like cubes and pyramids, which are bound by a number of flat faces, and where each face is bounded by a number of straight edges. The technical term for these solids is ‘polyhedra’. They’re the three-dimensional analogues of polygons.

A cube has 8 vertices, 6 faces and 12 edges and 8 + 6 – 12 = 2, so it works for a cube. In fact it works for any polyhedron. But what’s this got to do with our two-dimensional problem? It’s amazing the way a mathematician is able to change the subject. Talk to him (or her) about one thing and the next thing you know he or she is talking about something completely different and apparently unrelated.

But the true mathematical approach is to draw together the original problem and the apparent ‘red herring' and to show that they're very much related after all.

Take a polyhedron, for example a cube, and remove one of the faces. Now stretch the rest out so that it lies flat. You might need to use your imagination for this because a cardboard cube is not sufficiently elastic. The edges of the faces may no longer be straight. That doesn't matter. The important thing is which point is joined to what. Nothing in that department has changed; only the layout which now lies in a plane.



What we've produced is a 'map' with vertices, edges and faces (except that the faces would now be better described as regions). The numbers of vertices, faces and edges has not changed throughout this imaginative flattening. What about the face we removed to open it all up? Well that's just become the outside region of the map.

It's because V + F – E = 2 works for maps that it also works for polyhedra. But why does it work for maps? Well can we put that one on hold for a while. The best way to convince you is by a method called 'Mathematical Induction' and that's something we'll talk about later. Just be a good mathematical reader and accept it as fact. (Of course a really good mathematical reader will say “well, just for now, but eventually I want to know why”.) But to give you enough faith to keep you going, draw a few maps and check it out. A few confirming examples is no proof, but they’re comforting nevertheless!

Well back to our supposed solution to the Utilities Puzzle. We know that $V = 6$ and $E = 9$. We counted them. We have to have that number of vertices and that number of edges in any solution to the problem. Conceivably the number of faces, or regions could vary. But no. Euler's formula says that

$$F = E + 2 - V = 9 + 2 - 6 = 5.$$

So, indirectly, we can infer that any solution to the puzzle must have exactly 5 regions. Where does this get us?

The question to ask at this point is “what is the average number of edges per face?” Why this question? What led to asking that? That’s where mathematicians get really sneaky. Often it’s just a matter of asking the right question and it all falls out. So how does a mathematician develop the art of asking just the one question that will unravel a problem?

The answer is two-fold. Firstly, a mathematician, thinking about a certain problem, develops his or her intuition so that the ‘right’ question just pops out. It’s a common experience in the trade that after getting nowhere with a problem a mathematician puts it away and “sleeps on it”. Then suddenly the answer, or at least the right question which leads to the answer, comes as if from nowhere. He might be on a bus, she might be out walking. The problem is miles away. Then like a bolt from the blue, it comes.

The other explanation for why mathematicians seem to have this uncanny ability to hit on exactly the right question first time, is that they generally don’t. You see, in practice a mathematician might ask dozens or hundreds of questions about the problem in hand. Scores of screwed up sheets of paper might litter the floor until finally “eureka” – the right one comes.

Now you don’t think a mathematician is going to unravel all those crumpled-up pieces of paper and write up the whole investigation, false starts and all – of course not. You’d never want to read them and nor would any other mathematician. Only the right question, the right way of looking at the problem, gets into print.

It appears to the reader that Euclid, Euler or Einstein just sat down one day and wrote a theorem as effortlessly as, we’re told, Mozart wrote his music — flawless in first draft. All the pain and tears and sweat and sleepless nights and countless cups of coffee and conversations are hidden. All that appears is the finished product.

By the way, while we are talking about Einstein I should point out that, unlike what many people believe, he wasn’t a great mathematician. He was indeed a great theoretical physicist – probably the greatest that ever lived. And he was he had a good knowledge of mathematics, otherwise he couldn’t have applied it so well. But frequently he had to ask his colleagues about some difficult mathematical technique. And he never discovered any new mathematics. But a great theoretical physicist, yes.

Mathematicians are not the only ones who remove their scaffolding before displaying their finished edifice. But probably the process by which they achieve their results is less well understood than most.

What has all this to do with the problem in hand? Not a great deal. This was a real digression. If you remember, we had a map with 6 vertices, 9 edges and 5 regions. Well, we *supposed* we had such a map, because a solution to the Utilities Puzzle requires such a map to exist. And we were about to ask the RIGHT question. And this is ...

What is the average number of edges per face?

Easy! With 9 edges and 5 faces or regions the average number of edges per face is 9/5 = 1.8. Whoops! That's a bit on the low side! Think again!

Silly us. We forgot that each edge (boundary) separates *two* regions. Imagine that each edge is neatly sliced lengthwise into two half-edges. Now each half-edge is attached to only one region. Start again.

We have 9 edges, that’s 18 half-edges, to be shared among 5 regions. That’s 18/5 = 3.6 edges per face on average. That’s better. But maybe still a wee bit too small.

Each face has to have at least four edges. Why not 2? Well, that would mean two edges connecting the same two vertices and the puzzle specifies only one. Why not 3? Well a closed path has to alternate between utility and house. Three edges just wouldn’t work.

So if 4 is the smallest number of edges surrounding any one face then the average must be at least 4? An average below 4 is just not possible. In fact it’s *impossible*. Yet that impossible state of affairs is forced upon us if we assume that a solution exists. Therefore no solution can exist.

You see the infinitely many possibilities can be captured by a little piece of elementary arithmetic that in the end depended on the undeniable fact that 3.6 is less than 4. And all because we asked the right question!

§2.9. Is it possible to get GODEL from LODGE?

It would be understandable if you felt that you'd have enough impossibilities for now. By all means skip to the next chapter, taking a detour via the radio play *There Is No Time*. But if you're a glutton for algebraic punishment then read on.

The word GODEL (the logician whose work shook the foundations of mathematics to the core in the 1930's was actually Gödel, but we'll drop the umlaut over the "o") and the word LODGE use the same letters, so a simple rearrangement will do the job of getting GODEL from LODGE.

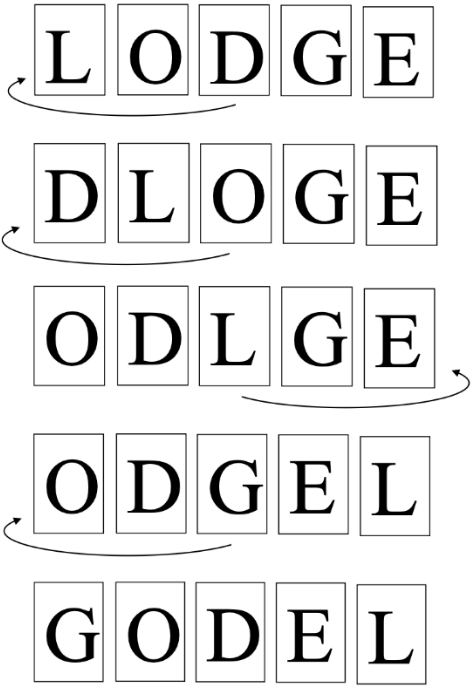
But suppose that the five letters are written on five cards and arranged in a row to spell LODGE, and suppose that a rule is imposed on how the cards are to be rearranged. Suppose that we're only allowed to move the middle card to either end, moving them up to close the gap. If that is all we're allowed to do, can we still get GODEL from LODGE?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

This is one of many puzzles that involve permutations, or rearrangements. The Rubik's Cube$^{\text{TM}}$ is perhaps the most famous, and probably the most complicated. There is another puzzle which was in vogue many years ago, called the “Fifteen Puzzle”. It consisted of fifteen small square tiles that could slide around in a 4 by 4 square frame.

With a permutation puzzle there are a number of pieces that can be moved and one or more possible moves that are permitted. In most cases the restrictions as to what rearrangements are allowed are automatically imposed by the engineering of the puzzle. In our LODGE-TO-GODEL puzzle, however, we’ve artificially imposed a restriction.

Well, this puzzle is a pretty easy one to solve:



Not very interesting. (If you want a harder challenge, try converting LEDOG to GODEL by the same rules.)

But suppose we introduce some different rules. There are two possible moves under these new rules.

* TRANSFER AN END CARD TO THE OTHER END
* REVERSE THE ORDER OF THE CARDS

There are just three basic moves:

move	**description**	**effect on LODGE**
L	take the left card and transfer it to the right	ODGEL
R	take the right card and transfer it to the left	ELODG
V	reverse the order of the cards	EGDOL

Using these new rules, can we get from OGLED to GODEL? With a bit of experimenting we see that we can:

OGLED → ODELG (move L) → DELGO (move L)
→ OGLED (move V).

Now, can we get from LODGE to GODEL? The answer is "NO". It is IMPOSSIBLE.

To see this, first observe that we can do without R, because R is the same as doing L four times, which we write as L^4. Anything that can be achieved using L, R and V can be achieved using just L and V alone.

Next, it is obvious that doing L five times brings us back to where we were. We write the operation of doing nothing by the symbol I and so we write $L^5 = I$. Similarly $V^2 = I$. Reversing the order twice in a row gets us back to where we started.

We can write a sequence of moves as a product of powers of L and V, but we only need powers of L up to 4, and we don't need any powers of V, other than V^1 which, of course, is just V.

Suppose you came up with a recipe for transforming LODGE into DOLEG as:

$$\mathbf{L^8V^3L^2VL^4V^5L^7}.$$

You might be tempted to collect all the L's together, and all the V's and write this is $L^{19}V^9$ and then simplify this to just LV. But notice that this doesn't work since V changes LODGE into EGDOL.

You can't bring the L's together and the V's, like you would in ordinary algebra. These moves don't commute, that is, $VL \neq LV$. If you start with LODGE, the move VL would turn it into GDOLE, while LV makes it LEGDO.

I would point out that L^8 is the same as L^3, because $L^5 = I$, and V^3 is the same as V and so this would simplify your solution to $\mathbf{L^3VL^2VL^4VL^2}$. Can we simplify it further?

Notice that VL = L^4V. Check it out. For example, VL turns LODGE into GDOLE.

L^4 turns LODGE into ELODG and V then changes this to GDOLE, which is the same as VL. This means that we can move a V across L's but as VL = L^4V so each L on the right of a V becomes L^4.

So your solution can be written as:

$L^3L^8VVL^4L^8V = L^{11}L^{12}V = L^{23}V = \mathbf{L^3V}$.

The fact that VL = L^4V means that any recipe can be written as L^mV^n. Since $m = 0, 1, 2, 3$ or 4 and $n = 0$ or 1, we only get 10 possible results from applying these rules:

I	**L**	**L^2**	**L^3**	**L^4**
LODGE	**ODGEL**	**DGELO**	**GELOD**	**ELODG**

V	**LV**	**L^2V**	**L^3V**	**L^4V**
EGDOL	**LEGDO**	**OLEGD**	**DOLEG**	**GDOLE**

Since GODEL is not one of these it cannot be achieved.

So, in the end, it came down to checking possibilities. But the breakthrough came when we realised that the infinitely many possibilities came down to just 10. This is a common situation in mathematics. Something, with infinitely many possibilities, is proved to be impossible by reducing these infinitely many possibilities to a finite number.

A classic example of this is the celebrated Four Colour Theorem. It began as a question in 1852 when Francis Guthrie, who was drawing and colouring a map of the counties of England, wondered whether four colours are enough. Guthrie had studied under the mathematician Augustus de Morgan at University College in London and his brother, Frederick was then studying mathematics under de Morgan. So Francis passed on the question to de Morgan through his brother.

De Morgan wrote: “A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured – four colours may be wanted but not more – the following is his case in which four colours are wanted. Query: cannot a necessity for five or more be invented.

Over the next few decades it became the Four Colour Conjecture. Many ‘proofs’ were published and several of them stood for a number of years before they were shown to be wrong. It took over 100 years before, in 1976, it was finally proved by Appel and Haken.

But the proof caused a lot of controversy in that it was the first theorem in history that was proved by a computer program. Of course no computer program could consider the infinitely many possible maps. What Appel and Haken did was to use standard mathematical reasoning to reduce this to 1,834 maps. If all these could be 4-coloured then every map could be 4-coloured. Here is the principal of reducing a proof of impossibility to checking a finite number of cases. However this time, the number of cases was rather large and required a computer to check them all. A computer program, laboriously, considered each of these maps and, indeed, showed that every one of them was 4-colourable.

Appel and Haken were at the University of Illinois when they published their proof and the local postal authorities were so proud of this discovery that for many years they franked letters that passed through their hands with the words FOUR COLORS SUFFICE. Indeed they were still using this slogan in 1994 as this picture shows.

Motivational speakers often use the slogan:

NOTHING IS IMPOSSIBLE!

I hope that, as a result of reading this chapter, you will realise that it is not really true. That slogan should read:

SOME THINGS ARE IMPOSSIBLE BUT THEY ARE LESS COMMON THAN YOU THINK!

INTERLUDE: RADIO SCRIPT

“It is Impossible – There is no Time”

Narrator: Mathematics and sport have this in common. They’re both a young man’s occupation. An historian reaches his peak in his sixties, an engineer at forty. A mathematician is said to be already on the decline at the age of thirty. Évariste Galois made his important discoveries in the theory of algebraic equations at the age of nineteen. At twenty he was dead.

Female Voice: Poor boy. What did he die of?

Narrator: He was killed in a duel.

Female Voice: Sounds like a character out of one of the Alexander Dumas novels.

Narrator: Almost. Alexander Dumas knew him and referred to him in one of his memoirs.

Female Voice: So Galois had to defend his mathematics with his rapier?

Narrator: Well no. For a start it was a duel fought with pistols, not swords. And secondly it was over a woman.

Female Voice: Just like a Frenchman!

Narrator: Perhaps. But in fairness I should point out that it was more than likely that she had been planted by his political adversaries to provide the excuse for a duel. It was really all to do with politics. You see, Galois had been very active in Republican politics and several times landed himself in trouble with the police. In fact much of his mathematics was done during spells in gaol. (*With feeling*) He wasn't afraid of death and he'd gladly have died for the Republican cause. But such glory was not to be.

Galois: I beg patriots and my friends to forgive me that in dying I do not die for my country. I die the victim of an infamous coquette. My life is quenched in a miserable piece of slander.

Oh, why do I have to die for such an unimportant cause; to die for something so contemptible? Farewell! It was my wish to give my life for the public good. Forgiveness to those who kill me. They are of good faith.

Narrator: These were the words he wrote to his friends on the night before the duel. He seemed quite sure that this night would be his last. He sat up all night writing some personal letters and then going over his mathematical papers. Scrawled across one of them he wrote the pathetic words ....

Galois: (*in despair*) I have no time.

Narrator: So many of his ideas had yet to be written down and there was just not enough time. He wrote ...

Galois: I hope some people will find it to their advantage to decipher all this mess.

Narrator: The duel took place on Wednesday 30th May 1832 just outside Paris. Galois was wounded and left lying by the roadside. Even his seconds deserted him. He was eventually found by a Good Samaritan and taken to hospital. It was in vain, for the next day he died.

His mathematical discoveries however were to lie on the roadside for a further eleven years. Not by the side of the road out of Paris but by the side of the highway of mathematical research. The Good Samaritan who rescued them was Joseph Liouville who in 1843 drew Galois' work to the attention of the French Academy.

Liouville: I hope to interest the Academy in announcing that among the papers of Évariste Galois I have found a solution, as precise as it is profound, of this beautiful problem: whether or not a given polynomial equation is soluble by radicals.

****

Narrator: What was this theory that was “beautiful” and “as precise as it is profound”? It was in fact the culmination of over two thousand years of mathematical enquiry into the theory of polynomial equations.

Most people have heard of quadratic equations. Most people vaguely remember what they are. Roughly speaking they're equations involving x^2. Maybe you also remember that there’s such a thing as a quadratic equation formula. Now I’m not expecting you to remember it — simply to know that it exists. It’s a formula into which you put the numbers from the equation, do some arithmetic, and out pop the answers.

The arithmetic isn’t hard, but at one stage it involves finding a square root. “Radical” means the same as “root”, and solving a polynomial equation by radicals simply means finding a formula for such an equation into which you plug the numbers from the equation and do some arithmetic, including finding square roots, cube roots or whatever roots may be necessary.

Now the Babylonians could do it for quadratic equations. In the sixteenth century the Italians worked out how to solve the cubic (involving powers of x up to x^3) and the quartic (powers up to x^4). These formulae are much more complicated than the one for the quadratic but they have a similar structure.

The next step should have been the quintic (powers up to x^5). But no such formula was forthcoming for the next three centuries. Finally in 1824 a 22 year-old Norwegian mathematician, Abel, called off the search – he proved that no such formula can possibly exist.

Abel's methods, however, were not very enlightening. They worked but they didn't make one feel that one knew why they worked. The methods of Galois a few years later were much more general and much more enlightening. Moreover he took the problem a stage further.

So, Abel has shown that there is no general formula for *all* polynomials involving x^5. But there *are* formulae that work for *some* of them. Which ones? Galois worked out exactly which ones are soluble by radicals and which ones are not.

Now make sure you understand what is being claimed. Not that some polynomial equations have no solutions. Solutions can be proved to exist, even if we can't find them. Not even that we can't find the solutions for practical purposes. There are methods, implemented by computers, which can find any solution to any degree of accuracy. It's a question of which polynomials can be solved *exactly*, by means of a *formula* involving radicals or roots.

Galois showed that corresponding to every polynomial equation is something called a group. And one can tell from the structure of this group whether or not the polynomial is soluble by radicals.

Now I think that rather than give you a formal, precise and technical definition of a group it would be better if I gave a broad and vague description, and then a specific example.

A group is a certain type of mathematical system where the things in it can be combined like multiplication. But the things needn't be numbers and the method of combination needn't be ordinary multiplication.

Pretty vague isn't it? Well I didn't want to get too technical. Now here's an example. It's called the "dihedral group of order 8". It crops up in many different guises. I could describe it to you the way Galois would have, in terms of substitutions of solutions of a certain polynomial equation, or, as it is presented in a modern course on Galois Theory, as automorphism groups of field extensions. But I won't. That's too hard.

Instead, let me dress it up as a children's party game. I've called it "duels" in memory of Galois. It's rather a fun sort of game that can be counted on to keep a bunch of bored children amused - for a few minutes anyway. Who said mathematics can't be useful!

"Duels" is a game basically like "O'Grady Says" where players are "out" if they make a mistake in obeying the leader's instructions.

The instructions are RIGHT, LEFT and LOAD. The instructions RIGHT and LEFT require you to turn through 90 degrees, left or right and to LOAD, you hold your hand up with two fingers outstretched as if holding a pistol. But here's the catch.

Whenever the gun is loaded you must do the opposite to what you are told.

If your gun is loaded and you're told to load, you must unload, that is, fire. And if told to turn right with a loaded gun you must turn left and vice versa. But only when the gun is loaded do you do the opposite. At other times you must obey the instructions exactly.

It's quite hilarious to watch when a number of people are playing and you really need to keep your wits about you to play it well.

Would you like to try it out right now? If you don't feel like standing up and obeying the instructions overtly you can remember which wall you're supposed to be facing, and discreetly raise your right hand whenever the gun is loaded.

Choose a particular starting direction as your "home" direction. Gun unloaded. Ready?

RIGHT
LOAD
RIGHT

Did you remember to obey this second right turn by turning left?

LOAD

You should once again be in your home position with your hands by your side having just fired the pistol.

Now there are eight positions you can be in during this game — four directions, each with a loaded or unloaded pistol. And there are basically eight different sets of instructions for getting you there.

We say that two sets of instructions are equal if they result in the final positions. So for example, LOAD LOAD LOAD would be the same as LOAD. (Never mind that in the first case you've fired a shot.) And three right turns would equal one left turn.

So you see, we've a mathematical system here consisting of eight things. The things aren't numbers – they're sets of instructions. And we can combine them like multiplication by doing one set of instructions after the other.

We get equations like:

RIGHT times RIGHT times RIGHT equals LEFT
and
RIGHT times LEFT equals LEFT times RIGHT

Now here's the interesting thing about this group which makes it quite different from groups of numbers. Are you in your starting position? Gun unloaded?

RIGHT
LOAD

I want you to remember which way you're facing. You just performed RIGHT times LOAD. Now go back to your home position, gun unloaded, and this time do

LOAD
RIGHT

that is, do the same two operations in reverse order. Notice that you've ended up in the opposite direction to before.

RIGHT times LOAD is not equal to
LOAD times RIGHT

We have what is called a **non-commutative** group.

The difference between commutative and non-commutative groups is very important in Galois Theory. Commutative groups are those where x times y is always equal to y times x. The dihedral group of order 8 is *non-commutative*.

Suppose you think of solution by radicals as a sort of “abstract stomach” and commutative groups as particles which can be absorbed by the stomach lining. Any group which can be broken up into commutative bits would therefore be digestible. The dihedral group, for example, can be broken into two commutative bits in a way that I won’t attempt to describe.

Galois showed that these digestible groups (or “soluble groups” as he called them) – these groups which can be broken down into commutative bits – are precisely the groups that correspond to polynomial equations that are soluble by radicals.

Some polynomial equations ($3x^5 - 5x^3 + 1 = 0$ for example) correspond to groups which are not soluble, or to use our analogy, they are indigestible. They involve a non-commutative chunk which cannot be broken down further. These polynomial equations are therefore not soluble by radicals. Not even by Galois, who was something of a radical in the political sense.

It was quite an achievement for a young man who only scribbled his mathematics in his spare time and threw the major part of his energies in fighting for the freedom of his country. On the eve of his duel he wrote to two of his friends ...

(*The sounds of “Le Marseillaise” are heard in the background.*)

Galois: I have been provoked by two patriots and it is impossible for me to refuse.

Your task is simple. I want to let it be known that I am fighting against my will after having exhausted all means of reconciliation. Please remember me, since fate did not allow me a life that would make my name worthy to be remembered by my country.

I die your friend,

É. Galois

(*The music of "Le Marseillaise" swells and reaches its dramatic conclusion.*)